SYNOPSIS OF THE THEORY OF FUNDAMENTAL EVOLUTION EQUATIONS

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To the **105th anniversary** of the founding of the Mathematical Institute of the Ukrainian Academy of Sciences

and to the **100th anniversary** of quantum theory

Kyïv, April 14, 2025

Outline

1. **I: Chronicle of IM for 105 years**
2. **II: A brief chronology of the 350-year-old theory of the fundamental evolution equations**
3. **III: The scales of the Universe's structures**
   - The megascale of the Universe ($10^{28}m \Leftrightarrow 10^{5}m$)
   - The macroscale of the Universe ($10^{5}m \Leftrightarrow 10^{-5}m$)
   - The mesoscale of the Universe ($10^{-5}m \Leftrightarrow 10^{-10}m$)
   - The microscale of the Universe ($10^{-10}m \Leftrightarrow 10^{-35}m$)
   - Time scales of the Universe ($4.35 \cdot 10^{18}s \Leftrightarrow 5.4 \cdot 10^{-44}s$)

I. CHRONICLE OF IM FOR 105 YEARS

II. A BRIEF CHRONOLOGY OF THE 350-YEAR-OLD THEORY OF THE FUNDAMENTAL EVOLUTION EQUATIONS

The 350-year-old theory of the classical fundamental evolution equations

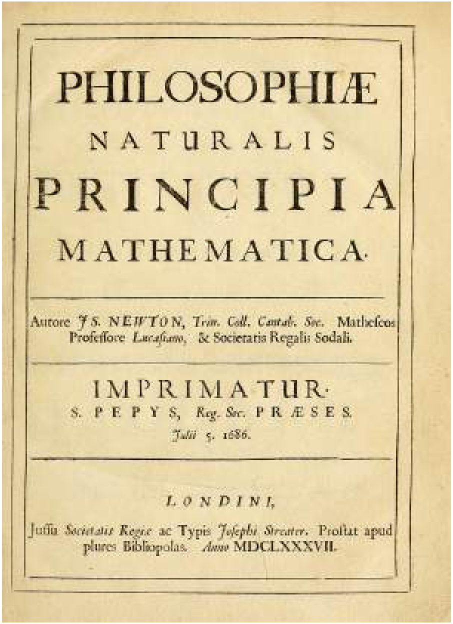
1687 Isaac Newton
"Philosophiæ Naturalis Principia Mathematica" (1687)

1788 Giuseppe Lodovico Lagrangia
"Mécanique analytique" (1788)

1809 Siméon Denis Poisson
"Traité de mécanique" (1811)

1827 William Rowan Hamilton
"On a General Method in Dynamics" (1834, 1835)

1838 Joseph Liouville
"Note sur la Théorie de la Variation des constantes arbitraires" (1838)



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *JS.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

The 100-year-old theory of the quantum fundamental evolution equations

1925 Werner Karl Heisenberg

1925, 1926 Paul Adrien Maurice Dirac

1926 Wolfgang Ernst Pauli

1926 Erwin Rudolf Josef Alexander Schrödinger

1927 John von Neumann

The 150-year-old theory of the many-particle evolution equations

1860, 1867 James Clerk Maxwell

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1900, 1912 David Hilbert

In Hilbert's own words: *"Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes which lead from the atomistic view to the laws of motion of continua*".

1902 Josiah Willard Gibbs

"Elementary principles of statistical mechanics" (1902)

1945 Mykola Mykolayovych Bogolyubov

"IM report" (1945)

80th Anniversary of the Bogolyubov hierarchy (BBGKY hierarchy)

The 160-year-old theory of the fundamental evolution equations of quantum fields

1865 James Clerk Maxwel

1915 David Hilbert

1925, 1928 Paul Adrien Maurice Dirac

1926 Oskar Klein, Walter Gordon, Vladimir Fock

1954 Chen-Ning Franklin Yang, Robert Laurence Mills

The fundamental evolution equations of classical mechanics. Newton – Lagrange equation (1687, 1788)

$$\frac{d^2}{dt^2}q(t) = -\frac{d}{dq(t)}\Phi(q(t)),$$

$$q(t)_{|t=0} = q^0,$$
$$\frac{d}{dt}q(t)_{|t=0} = p^0$$

Examples:
$\Phi(q) = 0$ – motion by inertia (straightforward, uniform);
$\Phi(q) = \frac{q^2}{2}$ – harmonic oscillator;
$\Phi(q) = \frac{1}{q}$ – conic sections – ellipse, parabola, hyperbola (empirical Kepler's laws of planetary motion);
$\Phi(q) = \begin{cases} \infty, & q < \sigma, \\ 0, & q \geq \sigma, \end{cases}$ – hard spheres ...

Hamilton equations (1827)

$$\frac{d}{dt}q_i(t) = \frac{\partial H_N}{\partial p_i(t)},$$
$$\frac{d}{dt}p_i(t) = -\frac{\partial H_N}{\partial q_i(t)},$$

$$q_i(t)_{|t=0} = q_i^0,$$
$$p_i(t)_{|t=0} = p_i^0$$

$$x_i \equiv (q_i, p_i) \in \mathbb{R}^3 \times \mathbb{R}^3, \ \ i = 1, \ldots, N$$

$$H_N \equiv H_N(x_1, \ldots, x_N) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j=1}^{N} \Phi(q_i - q_j)$$

The origins of the concept of the generator of fundamental evolution equations: Poisson brackets.

Simeon-Denis Poisson, 1809 ("A treatise of mechanics", 1842)

Carl Gustav Jacob Jacobi, 1862, 1842

Marius Sophus Lie, 1872

$$\{f_N, g_N\} \doteq \sum_{i=1}^{N} \left(\langle \frac{\partial f_N}{\partial q_i}, \frac{\partial g_N}{\partial p_i} \rangle - \langle \frac{\partial f_N}{\partial p_i}, \frac{\partial g_N}{\partial q_i} \rangle \right)$$

Anti-commutativity:

$$\{f_N, g_N\} = -\{g_N, f_N\},$$

Jacobi identity:

$$\{f_N, \{g_N, h_N\}\} + \{h_N, \{f_N, g_N\}\} + \{g_N, \{h_N, f_N\}\} = 0,$$

Linearity:

$$\{\alpha f_N + \beta g_N, h_N\} = \alpha\{f_N, h_N\} + \beta\{g_N, h_N\}, \quad \alpha, \beta \in \mathbb{R},$$

Leibniz rule:

$$\{f_N, g_N h_N\} = \{g_N, f_N\}h_N + g_N\{f_N, h_N\}$$

$$x_i(t) \equiv (q_i(t), p_i(t)) \in \mathbb{R}^3 \times \mathbb{R}^3,\ \ i = 1, \ldots, N$$

Canonical relations

$$\{q_i, p_j\} = \delta_{i,j},$$
$$\{q_i, q_j\} = 0,$$
$$\{p_i, p_j\} = 0$$

Hamilton equations (1842)

$$\frac{d}{dt} x_i(t) = \{x_i(t), H_N\}$$

Liouville equation (1838)

$$A_N(t) \equiv A_N(t, x_1, \ldots, x_N) = A_N^0(x_1(t), \ldots, x_N(t)),$$
$$x_i(t) \equiv x_i(t, x_1, \ldots, x_N)$$

$$\frac{\partial}{\partial t} A_N(t) = \{A_N(t), H_N\},$$
$$A_N(t)_{|t=0} = A_N^0$$

1. Time reversibility (Poincaré's reversion theorem);
2. Law of conservation of energy;
3. Invariance under the Galilean transformation (G. Galilei, 1638)

   Note: Lorentz transformation (H. Lorentz, 1892),

   Poincaré transformation (H. Poincaré, 1906)

The mean value functional (mathematical expectation) of observables

Riesz's theorem (1909) on the representation of a linear functional

$$\langle A_N \rangle = \left(\int d\mu(x_1, \ldots, x_N) \right)^{-1} \int A_N(x_1, \ldots, x_N) d\mu(x_1, \ldots, x_N),$$
$$d\mu(x_1, \ldots, x_N) = D_N(x_1, \ldots, x_N) dx_1 \ldots dx_N,$$

$$\langle A_N \rangle(t) \equiv (I, D_N^0)^{-1} (A_N(t), D_N^0) \equiv$$
$$(I, D_N^0)^{-1} \int A_N(t, x_1, \ldots, x_N) D_N^0(x_1, \ldots, x_N) dx_1 \ldots dx_N =$$
$$(I, D_N(t))^{-1} \int A_N^0(x_1, \ldots, x_N) D_N(t, x_1, \ldots, x_N) dx_1 \ldots dx_N,$$

$$D_N(t, x_1, \ldots, x_N) \doteq D_N^0(x_1(-t), \ldots, x_N(-t))$$

Observables and a state

Dual Liouville equation

$$\frac{\partial}{\partial t} D_N(t) = \{H_N, D_N(t)\},$$
$$D_N(t)_{|t=0} = D_N^0$$

$$\{H_N, D_N(t)\} \doteq \mathcal{L}_N^* D_N(t),$$
$$\{A_N(t), H_N\} \doteq \mathcal{L}_N A_N(t),$$
$$\mathcal{L}_N^* = -\mathcal{L}_N$$

$$(S_N(t)A_N^0)(x_1, \ldots, x_N) \doteq A_N^0(x_1(t), \ldots, x_N(t)),$$
$$(S_N^*(t)D_N^0)(x_1, \ldots, x_N) = D_N^0(x_1(-t), \ldots, x_N(-t)) \doteq (S_N(-t)D_N^0)(x_1, \ldots, x_N)$$

Pure and mixed states

Probability density: $(I, D_N(t))^{-1} D_N(t, x_1, \ldots, x_N) dx_1 \ldots dx_N$

$$D_N(t, x_1, \ldots, x_N) =$$
$$\int D_N(\tilde{x}_1, \ldots, \tilde{x}_N) \sum_{\{i_1, \ldots, i_N\}} \prod_{k=0}^{N} \delta(x_{i_k} - x_k(t, \tilde{x}_1, \ldots, \tilde{x}_N)) d\tilde{x}_1 \ldots d\tilde{x}_N,$$

where

$$\int D_N(\tilde{x}_1, \ldots, \tilde{x}_N) d\tilde{x}_1 \ldots d\tilde{x}_N = 1$$

The 100-year-old theory of the quantum fundamental evolution equations (a finite number of degrees of freedom)

1925 Werner Karl Heisenberg

1925, 1926 Paul Adrien Maurice Dirac

1926 Wolfgang Ernst Pauli

1926 Erwin Rudolf Josef Alexander Schrödinger

1927 John von Neumann

Observables and a state of quantum systems

Observables are self-adjoint operators $\hat{A}$ defined on a Hilbert space $\mathcal{H}$.

Examples of observables: $\hat{x}_i \equiv (\hat{q}_i, \hat{p}_i),\ i = 1, \ldots, N$ are defined on $\mathcal{H}_N \equiv \mathcal{H}^{\otimes N}$ (Maxwell-Boltzmann statistics)

$$
\begin{aligned}
&[\hat{q}_i, \hat{q}_j] \doteq (\hat{q}_i\hat{q}_j - \hat{q}_j\hat{q}_i) = 0, \\
&[\hat{p}_i, \hat{p}_j] = 0, \\
&[\hat{q}_i, \hat{p}_j] = i\hbar\hat{I}\delta_{i,j}
\end{aligned}
$$

The representation of canonical commutation relations in L^2_N

$$
\begin{aligned}
(\hat{p}_i\varphi)(\xi_1, \ldots, \xi_N) &= -i\hbar\frac{\partial}{\partial\xi_i}\varphi(\xi_1, \ldots, \xi_N), \\
(\hat{q}_i\varphi)(\xi_1, \ldots, \xi_N) &= \xi_i\varphi(\xi_1, \ldots, \xi_N)
\end{aligned}
$$

Examples: Hamiltonian of quantum systems

$$\hat{H}_N = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2} + \sum_{i<j=1}^{N} \Phi(\hat{q}_i, \hat{q}_j)$$

$$\hat{H}_N = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \Delta_{\xi_i} + \sum_{i<j=1}^{N} \Phi(\xi_i, \xi_j)$$

One-dimensional quantum oscillator

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} = -\frac{\hbar^2}{2}\frac{d^2}{d\xi^2} + \frac{\xi^2}{2} =$$

$$\hbar\, \mathfrak{a}^+ \mathfrak{a} + \frac{\hbar}{2}$$

Heisenberg equation (1925)

$$\hat{A}_N(t) = \mathcal{G}_N(t)\hat{A}_N^0,$$

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_N(t)\hat{A}_N^0 \doteq e^{\frac{i}{\hbar}t\hat{H}_N}\hat{A}_N^0 e^{-\frac{i}{\hbar}t\hat{H}_N},$$

$$\frac{d}{dt}\hat{A}_N(t) = \left[\hat{A}_N(t), \hat{H}_N\right],$$
$$\hat{A}_N(t)|_{t=0} = \hat{A}_N^0,$$

$$\left[\hat{A}_N(t), \hat{H}_N\right] \doteq -\frac{i}{\hbar}(\hat{A}_N(t)\hat{H}_N - \hat{H}_N\hat{A}_N(t)) \equiv \mathcal{N}_N\hat{A}_N(t)$$

The mean value functional (mathematical expectation) of observables of quantum systems

$$\langle \hat{A}_N \rangle = (\mathrm{Tr}\hat{D}_N)^{-1}\mathrm{Tr}\hat{A}_N\hat{D}_N,$$

the density operator $\hat{D}_N = D_N(\hat{x}_1, \ldots, \hat{x}_N) \equiv D_N(1, \ldots, N)$
is defined on the space $\mathcal{H}_N \equiv \mathcal{H}^{\otimes N}$
(the kernel of the operator $\hat{D}_N$ is known as the density matrix)

$$(\hat{D}_N\varphi)(\xi_1, \ldots, \xi_N) = \int d\xi'_1 \ldots d\xi'_N D_N(\xi_1, \ldots, \xi_N; \xi'_1, \ldots, \xi'_N)\varphi(\xi'_1, \ldots, \xi'_N)$$

$$\mathrm{Tr}_{1,\ldots,N} D_N(1, \ldots, N) = \int d\xi_1 \ldots d\xi_N D_N(\xi_1, \ldots, \xi_N; \xi_1, \ldots, \xi_N)$$

Von Neumann equation (1927)

$$\langle \hat{A}_N \rangle(t) \equiv (\mathrm{Tr}\, \hat{D}_N^0)^{-1} \mathrm{Tr}\, \hat{A}_N(t) \hat{D}_N^0 = (\mathrm{Tr}\, \hat{D}_N^0)^{-1} \mathrm{Tr}\, \mathcal{G}_N(t) \hat{A}_N^0 \hat{D}_N^0 = (\mathrm{Tr}\, \mathcal{G}_N^*(t) \hat{D}_N^0)^{-1} \mathrm{Tr}\, \hat{A}_N^0 \mathcal{G}_N^*(t) \hat{D}_N^0 \equiv (\mathrm{Tr}\, \hat{D}_N(t))^{-1} \mathrm{Tr}\, \hat{A}_N^0 \hat{D}_N(t)$$

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_N^*(t) \hat{D}_N^0 \doteq e^{-\frac{i}{\hbar} t \hat{H}_N} \hat{D}_N^0 e^{\frac{i}{\hbar} t \hat{H}_N}$$

$$\frac{d}{dt} \hat{D}_N(t) = \big[\hat{H}_N, \hat{D}_N(t) \big],$$
$$\hat{D}_N(t)|_{t=0} = \hat{D}_N^0$$

$$\big[\hat{H}_N, \hat{D}_N(t) \big] \doteq -\frac{i}{\hbar} (\hat{H}_N \hat{D}_N(t) - \hat{D}_N(t) \hat{H}_N) \equiv \mathcal{N}_N^* \hat{D}_N(t)$$

Pure and mixed states of quantum systems

$$\hat{D} = \sum\nolimits_k \lambda_k P_{\psi_k},$$

$$\lambda_k \geq 0, \; \sum\nolimits_k \lambda_k = 1, \quad P_{\psi_k}\varphi = (\varphi, \psi_k)\psi_k, \quad \psi_k, \varphi \in \mathcal{H},$$

$$(P_\psi \varphi)(\xi) \doteq \int d\xi' \psi(\xi)\psi^*(\xi')\varphi(\xi'),$$

$$\mathrm{Tr} P_\psi = \int d\xi \psi(\xi)\psi^*(\xi) = \int d\xi |\psi(\xi)|^2 = 1$$

$$P_{\psi_N}(t) = \mathcal{G}_N^*(t) P_{\psi_N} = P_{\psi_N(t)}, \quad \psi_N(t) = e^{-\frac{i}{\hbar} t \hat{H}_N} \psi_N$$

Schrödinger equation (1926)

$$i\hbar\frac{\partial}{\partial t}\psi_N(t) = \hat{H}_N\psi_N(t),$$
$$\psi(t)_N\big|_{t=0} = \psi_N^0$$

$$i\hbar\frac{\partial}{\partial t}\psi_N(t) = -\frac{\hbar^2}{2}\sum_{i=1}^{N}\Delta_{\xi_i}\psi_N(t) + \sum_{i<j=1}^{N}\Phi(\xi_i,\xi_j)\psi_N(t)$$

Note. Quantum oscillator (motion with discrete energy values). Ground state (vacuum) with energy $\frac{\hbar}{2}$.

Note. Some properties of quantum systems

Variance of the observables
$var(\hat{A}) = \left(\mathrm{Tr}(\hat{A} - \langle\hat{A}\rangle)^2\right)^{\frac{1}{2}}$

Heisenberg uncertainty relation
$var(\hat{A})var(\hat{B}) \geq \frac{\hbar}{2}|\langle[\hat{A},\hat{B}]\rangle|$

The quantization problem
$qp = pq = \frac{1}{2}(qp + pq) \to \hat{q}\hat{p}, \hat{p}\hat{q}, \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$

Systems of fermions or bosons. Spin of particles

$\mathcal{H}_N^{\pm} = \mathcal{S}_N^{\pm}\mathcal{H}^{\otimes N}$, where $\mathcal{S}_N^{\pm} = (S_N^{\pm})^2$ are the orthogonal projectors onto the symmetric subspaces $\mathcal{H}_N^{+}$ and the antisymmetric subspaces $\mathcal{H}_N^{-}$ of the tensor products of the Hilbert space $\mathcal{H}$.

Evolution equations of quantum systems: phase space representation

1927 Hermann Klaus Hugo Weyl
1932 Eugene Paul Wigner

$$(A(\hat{q},\hat{p})\varphi)(q) = \int dq' A(q,q')\varphi(q'),$$

Weyl transformation

$$A(q,p) = \int dq' e^{-\frac{i}{\hbar}\langle p,q'\rangle} A(q+\frac{q'}{2}, q-\frac{q'}{2})$$

$A(q,p)$ is the symbol of the Weyl operator $A(\hat{q},\hat{p})$.

Mean value functional

$$\mathrm{Tr}\, A(\hat{q},\hat{p})D(\hat{q},\hat{p}) = \hbar^{-1}\int A(q,p)D(q,p)dqdp$$

The pure state $D(q,q') = \psi(q)\psi^*(q')$ is the
Wigner distribution function

$$W(q,p) = \frac{1}{\hbar}\int dq' e^{-\frac{i}{\hbar}\langle (p,q')\rangle}\psi(q+\frac{q'}{2})\psi^*(q-\frac{q'}{2})$$

The generator of the von Neumann equation

$$\big(\mathcal{N}^* D\big)(q,p) = -p\frac{\partial}{\partial q}D(q,p) +$$
$$+\frac{i}{\hbar}\frac{1}{(2\pi)^3}\int dp' d\eta e^{i\langle (p-p'),\eta\rangle}\big(\Phi(q+\frac{\hbar}{2}\eta) - \Phi(q-\frac{\hbar}{2}\eta)\big)D(q,p'),$$

where $D(q,p)$ is the Weyl symbol of the operator $D(\hat{q},\hat{p})$.

Note. Quasi-classical limit (*correspondence principle*)

$$\lim_{\hbar\to 0}\big(\mathcal{N}^* D\big)(q,p) = \big(\mathcal{L}^* D\big)(q,p) = \{(\frac{p^2}{2} + \Phi(q)), D(q,p)\}$$

The 150-year-old theory of the many-particle evolution equations (an infinite number of degrees of freedom)

1860, 1867 James Clerk Maxwell

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In Hilbert's own words: *"Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes which lead from the atomistic view to the laws of motion of continua*".

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"Institute of Mathematics report" (1945)

80th Anniversary of the Bogolyubov hierarchy (BBGKY hierarchy)

The mean value functional (mathematical expectation) of observables

$$\langle A\rangle(t) \equiv (I, D(0))^{-1}(A(t), D(0)) = (I, D(t))^{-1}(A(0), D(t)),$$

$$A(t) = (A_0, A_1(t, x_1), \ldots, A_n(t, x_1, \ldots, x_n), \ldots),$$
$$(b, f) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \int b_n(x_1, \ldots, x_n) f_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

$$A^{(N)}(t) = (0, \ldots, A_N(t, x_1, \ldots, x_N), 0, \ldots),$$
$$(I, D^{(N)}(0)) = \frac{1}{N!} \int D_N^0(x_1, \ldots, x_N) dx_1 \ldots dx_N$$

$$\langle A\rangle(t) \equiv (I, D(0))^{-1}(A(t), D(0)) = (B(t), F(0)),$$

$$B_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{s} \frac{(-1)^n}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1}^{s} A_{s-n}(t, (x_1, \ldots, x_s) \setminus (x_{j_1}, \ldots, x_{j_n})),$$

$$F_s^0(x_1, \ldots, x_s) = (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int D_{s+n}^0(x_1, \ldots, x_{s+n}) dx_1 \ldots dx_{s+n}$$

$$(B(t), F(0)) = (B(0), F(t))$$

Note. $(\mathfrak{a} f)_n(x_1, \ldots, x_n) \doteq \int f_{n+1}(x_1, \ldots, x_n, x_{n+1}) dx_{n+1},$

$$(\mathfrak{a}^+ g)_n(x_1, \ldots, x_n) \doteq \sum_{j=1}^{n} g_{n-1}((x_1, \ldots, x_n) \setminus (x_j))$$

Hierarchies of evolution equations

$$\frac{d}{dt}B(t) = e^{-\mathfrak{a}^+}\mathcal{L}e^{\mathfrak{a}^+}B(t)$$

BBGKY hierarchy

$$\frac{d}{dt}F(t) = e^{\mathfrak{a}}\mathcal{L}^*e^{-\mathfrak{a}}F(t)$$

$$e^{\mathfrak{a}}\mathcal{L}^*e^{-\mathfrak{a}} = \mathcal{L}^* + [\mathfrak{a}, \mathcal{L}^*],$$

$$(\mathcal{L}^* + [\mathfrak{a}, \mathcal{L}^*]F(t))_s(x_1, \ldots, x_s) =$$
$$\{H_s, F_s(t)\} + \sum_{i=1}^{s}\int\{\Phi(q_i - q_{s+1}), F_{s+1}(t, x_1, \ldots, x_{s+1})dx_{s+1}\}$$

Kinetic equations
Equations of a continuous medium (hydrodynamic equations)

Boltzmann kinetic equation (1872)

$$\frac{\partial}{\partial t} f_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle f_1(t, x_1) + \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2_+} dp_2 d\eta \, \langle \eta, (p_1 - p_2) \rangle \big(f_1(t, p_1^*, q_1) f_1(t, p_2^*, q_1) - $$
$$ - f_1(t, p_1, q_1) f_1(t, p_2, q_1) \big)$$

Hydrodynamic equations

(Euler equations (1757), Navier–Stokes equations (1842), Hilbert expansion (1912))

The 160-year-old theory of the fundamental evolution equations (continuum of degrees of freedom)

1865 James Clerk Maxwel

J. Clerk Maxwell. A Dynamical Theory of the Electromagnetic Field. Phil. Trans. R. Soc. Lond. 1865 155, 459–512 (published 1 January 1865)

1915 David Hilbert, Albert Einstein

1925, 1928 Paul Adrien Maurice Dirac

1926 Oskar Klein, Walter Gordon, Vladimir Fock

1954 Chen-Ning Franklin Yang, Robert Laurence Mills

Phil. Trans. R. Soc. Lond. 1865 155, 459–512 (1 January 1865)


[459]

VIII. *A Dynamical Theory of the Electromagnetic Field.* By J. Clerk Maxwell, F.R.S.

Received October 27,—Read December 8, 1864.

PART I.—INTRODUCTORY.

(1) The most obvious mechanical phenomenon in electrical and magnetical experiments is the mutual action by which bodies in certain states set each other in motion while still at a sensible distance from each other. The first step, therefore, in reducing these phenomena into scientific form, is to ascertain the magnitude and direction of the force acting between the bodies, and when it is found that this force depends in a certain way upon the relative position of the bodies and on their electric or magnetic condition, it seems at first sight natural to explain the facts by assuming the existence of something either at rest or in motion in each body, constituting its electric or magnetic state, and capable of acting at a distance according to mathematical laws.

In this way mathematical theories of statical electricity, of magnetism, of the mechanical action between conductors carrying currents, and of the induction of currents have been formed. In these theories the force acting between the two bodies is treated with reference only to the condition of the bodies and their relative position, and without any express consideration of the surrounding medium.

These theories assume, more or less explicitly, the existence of substances the particles of which have the property of acting on one another at a distance by attraction or repulsion. The most complete development of a theory of this kind is that of M. W. Weber*, who has made the same theory include electrostatic and electromagnetic phenomena.

In doing so, however, he has found it necessary to assume that the force between two electric particles depends on their relative velocity, as well as on their distance.

This theory, as developed by MM. W. Weber and C. Neumann†, is exceedingly ingenious, and wonderfully comprehensive in its application to the phenomena of statical electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement, and by actually determining electrical quantities with an accuracy hitherto unknown.

* Electrodynamische Maassbestimmungen. Leipzic Trans. vol. i. 1849, and Taylor's Scientific Memoirs, vol. v. art. xiv.

† "Explicare tentatur quomodo fiat ut lucis planum polarizationis per vires electricas vel magneticas declinetur."—Halis Saxonum, 1858.

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The Maxwell equations

$$\frac{\partial F^{ik}}{\partial x^k} = 0,$$

$$F^{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

The Dirac equation

$$i\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} - m\psi = 0,$$

$$\overline{\psi} = \gamma^{0}\psi^{*}$$

The Klein–Gordon–Fock equation

$$(\Box + m^2)\varphi(x) = 0,$$

$$\Box = \partial_{x_0}^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$$

The Yang–Mills equations

$$\frac{\partial F^{\alpha}_{\mu\nu}}{\partial x^{\mu}} + f^{\alpha\beta\gamma} A^{\beta}_{\mu} F^{\gamma}_{\mu\nu} = 0,$$

$$F^{\alpha}_{\mu\nu} = \frac{\partial A^{\alpha}_{\nu}}{\partial x^{\mu}} - \frac{\partial A^{\alpha}_{\mu}}{\partial x^{\nu}} + f^{\alpha\beta\gamma} A^{\beta}_{\mu} A^{\gamma}_{\nu}$$

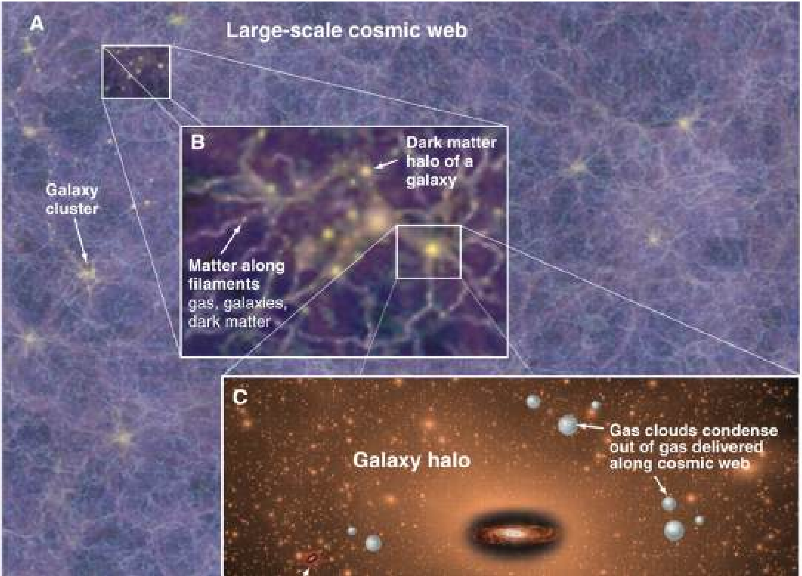
The gravity field equations

Einstein, A. (1915). Die Grundlage der allgemeinen Relativitätstheorie. Annalen der Physik, **49**(7), 769–822.

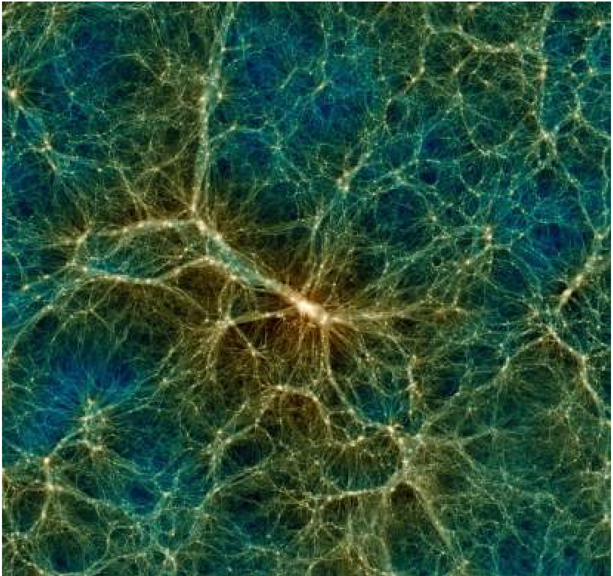
Hilbert, D. (1915). Die Grundlagen der Physik. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1915, 395–407.

III.1. THE MEGASCALE OF THE UNIVERSE ($10^{28} m \Leftrightarrow 10^5 m$)
The largest structures in the Universe

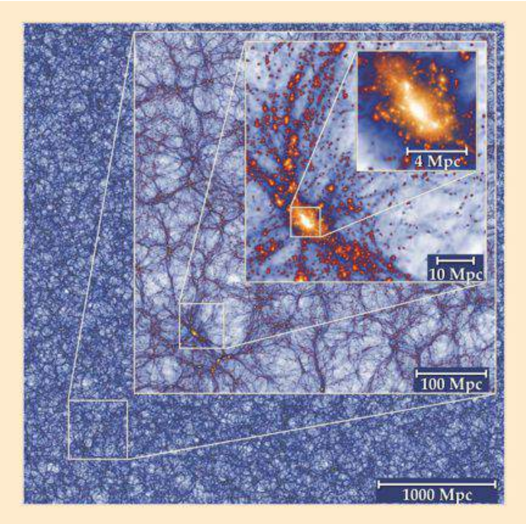
The cosmic web $\sim 2 \cdot 10^{25} m$



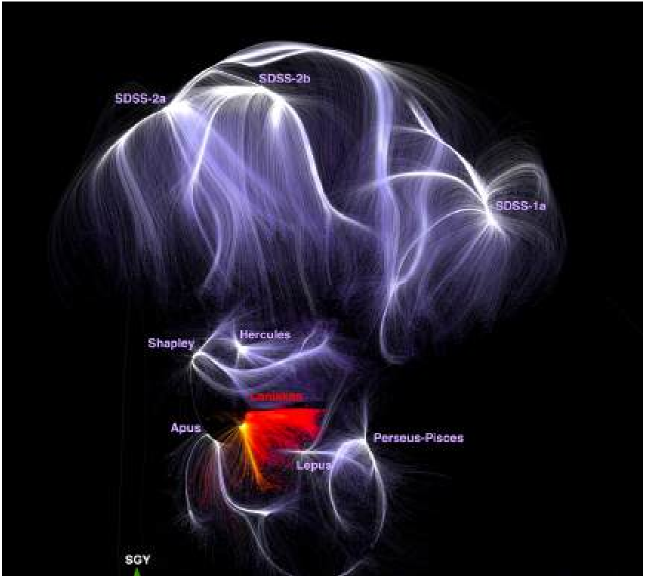
Massive filaments of galaxies separated by giant voids



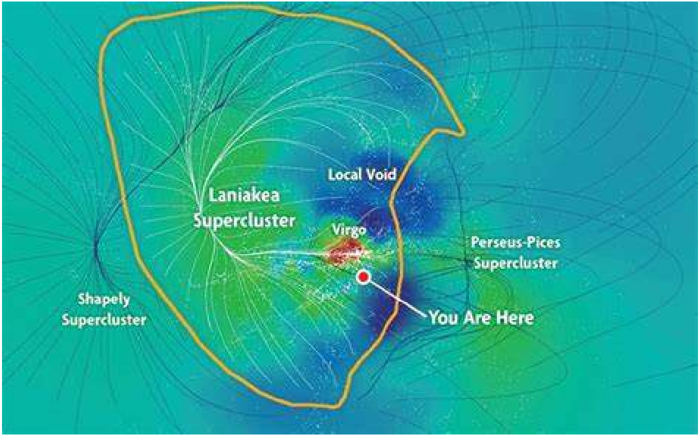
Note. The number of galaxies in the Universe $\sim 10 \cdot 10^{12}$ galaxies or $\sim 10^{24}$ stars



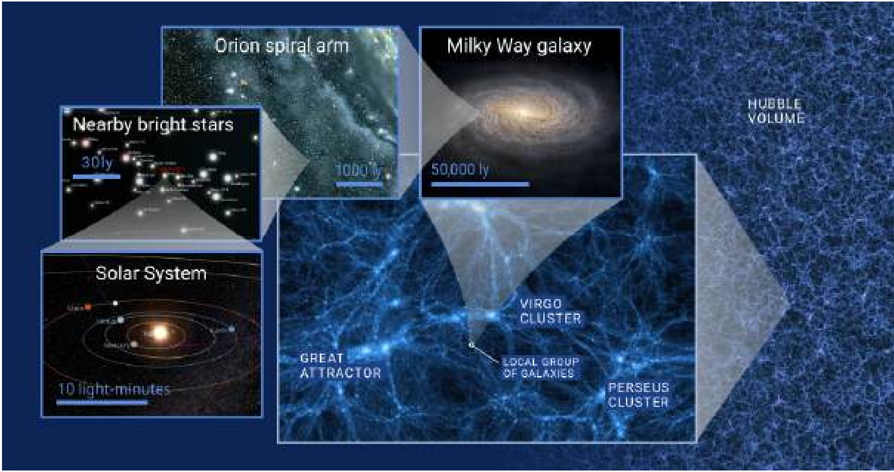
(scale $> 10^{24}m$)


SDSS-2b
SDSS-2a
SDSS-1a
Hercules
Shapley
Apus
Perseus-Pisces
Lepus
SGY

Laniakea Supercluster (scale $5 \cdot 10^{24}\, m$)



Virgo cluster ($1.5 \cdot 10^{23} m$), Milky Way ($1.2 \cdot 10^{21} m$)



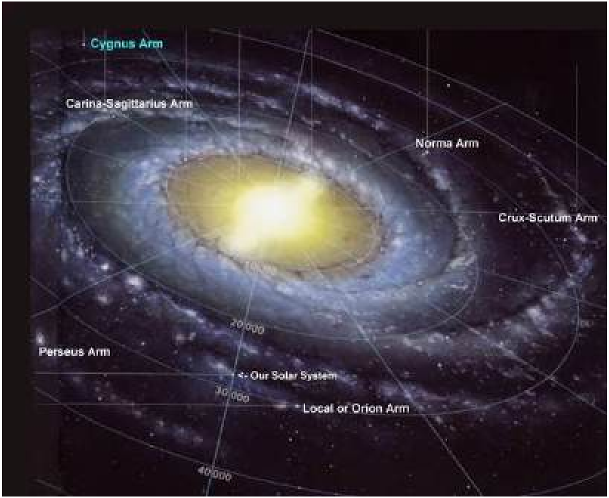
Note. 10^7 superclusters in the Universe

Scale $10^{21} \leftrightarrow 10^{24}m$

Note. The number of galaxies in the Universe $\sim 10^{13}$ ($\sim 10^{24}$ stars)

Andromeda galaxy ($1.5 \cdot 10^{21} m$)

Milky Way ($1.2 \cdot 10^{21} m$)



The megascale of the Universe ($10^{28} m \Leftrightarrow 10^5 m$)

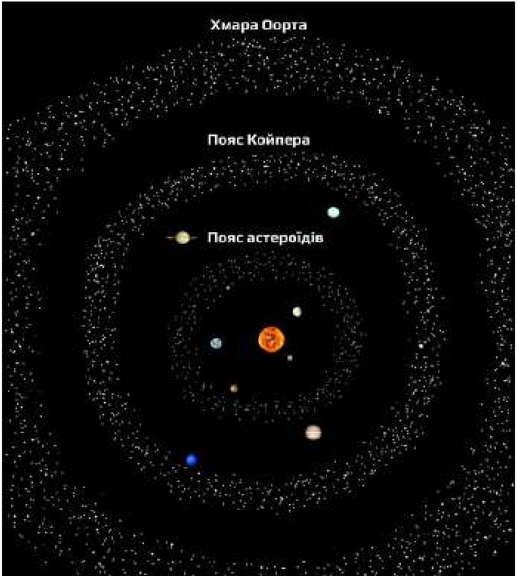
The megascale of the Universe ($10^{28}m \Leftrightarrow 10^5 m$)

Note. Stars in the Milky Way galaxy $\sim 10^9$

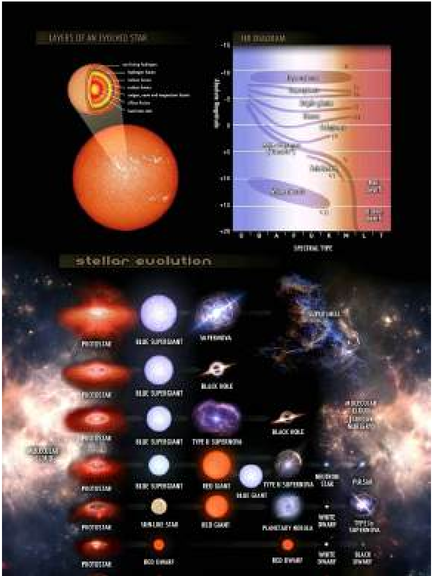
Scale $10^9 \leftrightarrow 10^{14} m$

Note. The number of black holes in the Universe $\sim 40 \cdot 10^{18}$

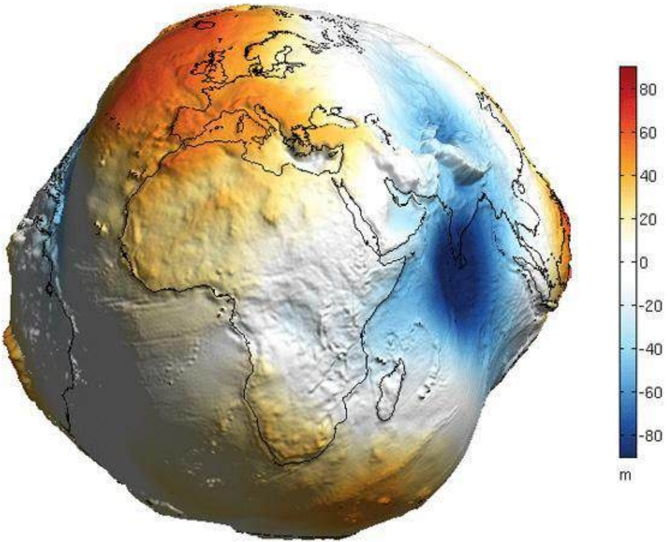
Scale $1.5 \cdot 10^{14}m$



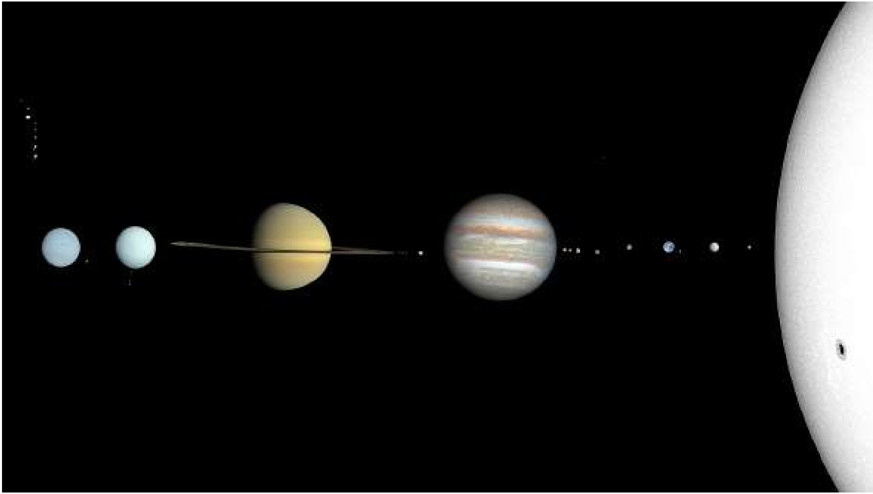
Stellar scale $2 \cdot 10^9 \leftrightarrow 3 \cdot 10^{12} m$



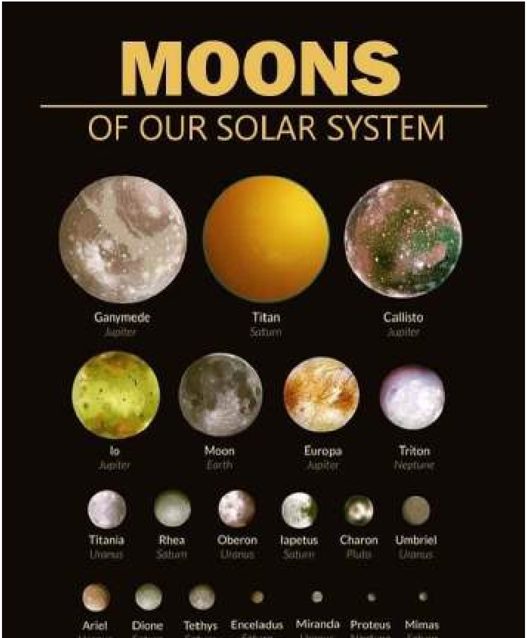
Earth ($10^7 m$)



The megascale of the Universe ($10^{28}m \Leftrightarrow 10^{5}m$)



The megascale of the Universe ($10^{28} m \Leftrightarrow 10^5 m$)



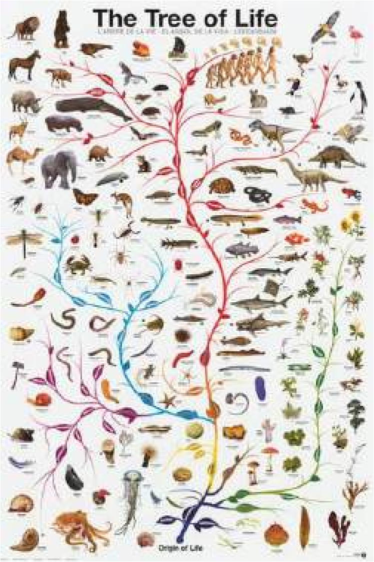
III.2. THE MACROSCALE OF THE UNIVERSE ($10^5 m \Leftrightarrow 10^{-5} m$)
Human civilization (scale $10^3 m$)

Scale $10^2 m$

Scale $10^2 m$

Scale $10^1 m$

$> 10^1 m$



Flora

Tea plant

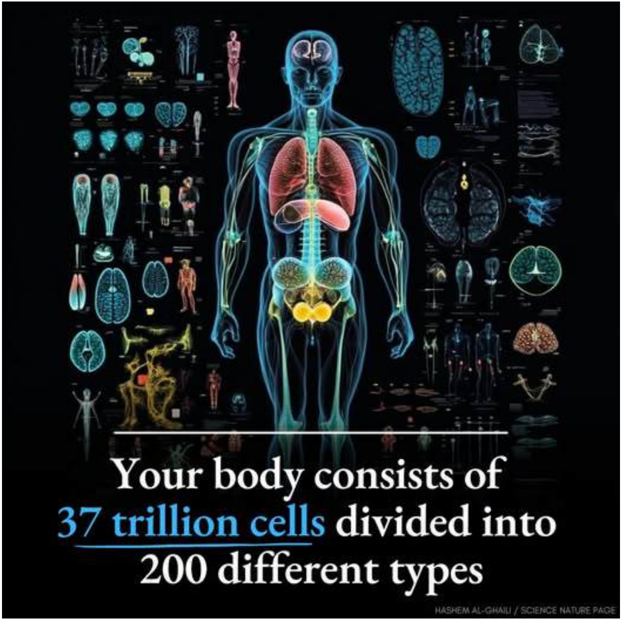
The macroscale of the Universe ($10^5 m \Leftrightarrow 10^{-5} m$)

Cherry Love

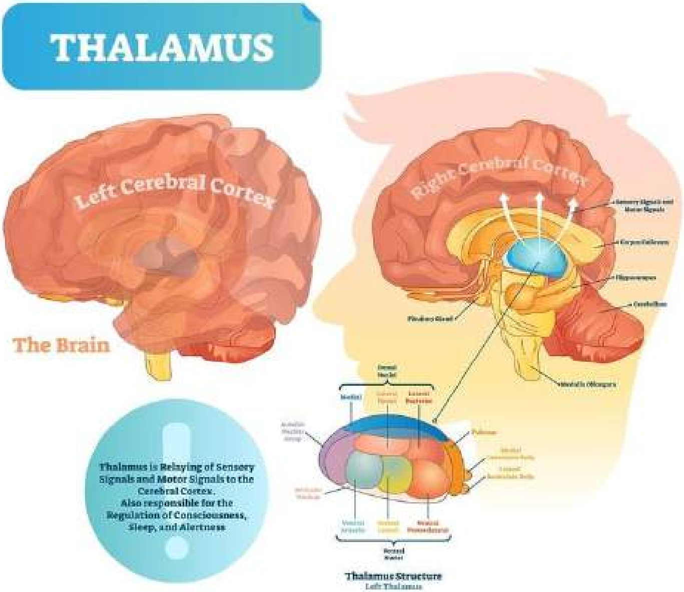
$> 10^1 m$

$10^1 m$

The macroscale of the Universe ($10^5 m \Leftrightarrow 10^{-5} m$)



The macroscale of the Universe ($10^5 m \Leftrightarrow 10^{-5} m$)



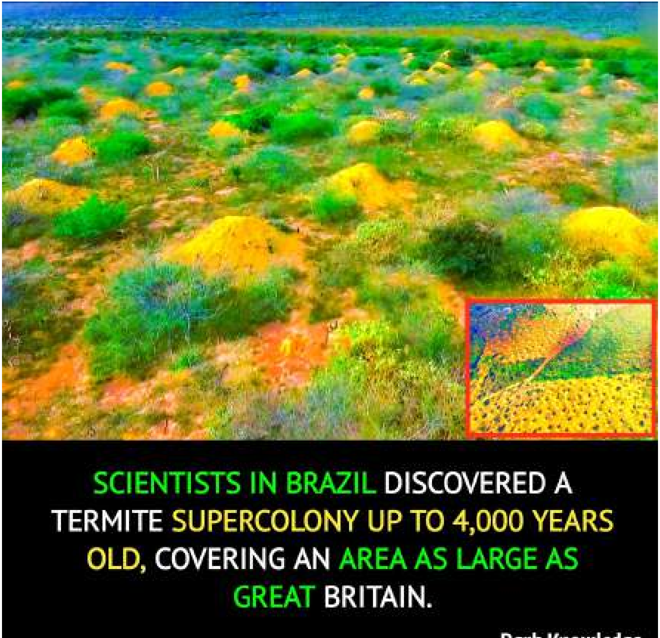
The macroscale of the Universe ($10^5 m \Leftrightarrow 10^{-5} m$)

Coral colonies $(2.5 \cdot 10^{-1} m \Leftrightarrow 10^{-3} m)$

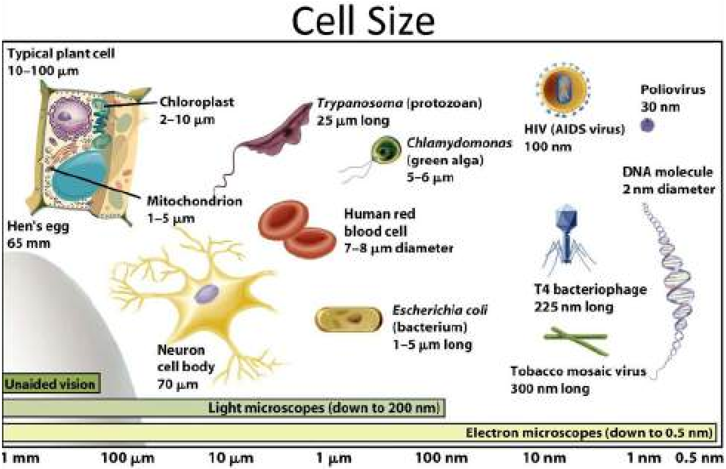
Kingdom of fungi $(10^{-1} m \Leftrightarrow 5 \cdot 10^{-2} m)$

Organized colonies of ants $(10^{-2} m \Leftrightarrow 5 \cdot 10^{-3} m)$

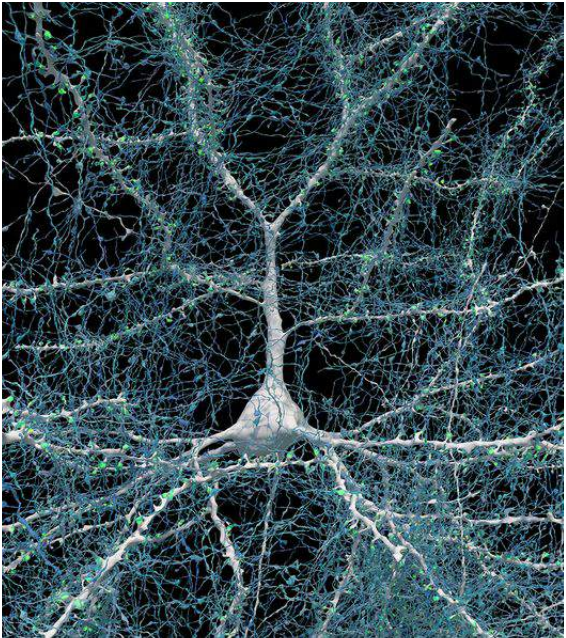
The macroscale of the Universe ($10^5 m \Leftrightarrow 10^{-5} m$)



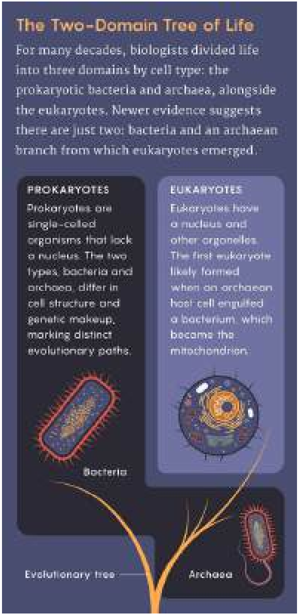
III.3. THE MESOSCALE OF THE UNIVERSE ($10^{-5}m \Leftrightarrow 10^{-10}$)
Cell size ($1\mu m = 10^{-6}m,\ 1nm = 10^{-9}m$)



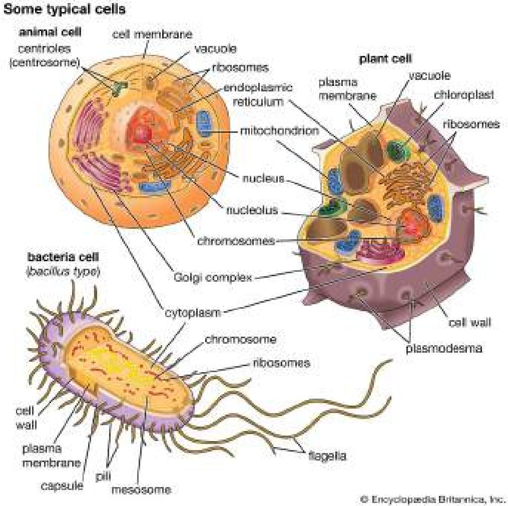
The mesoscale of the Universe ($10^{-5}m \Leftrightarrow 10^{-10}m$)



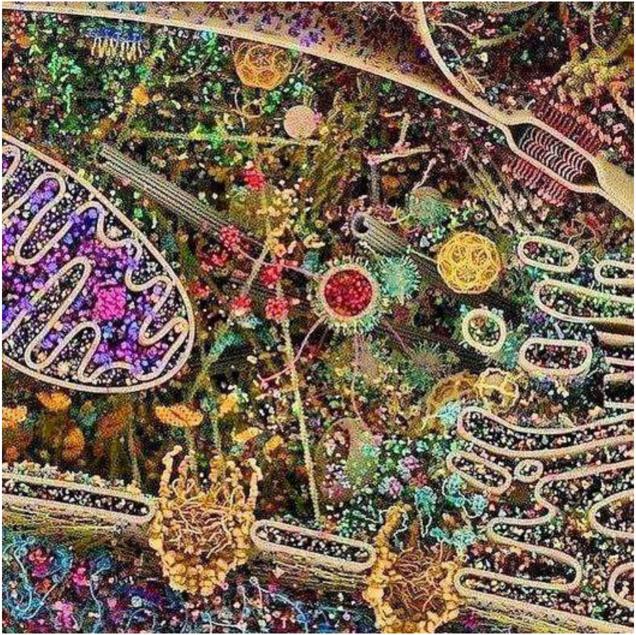
The mesoscale of the Universe ($10^{-5}m \Leftrightarrow 10^{-10}m$)



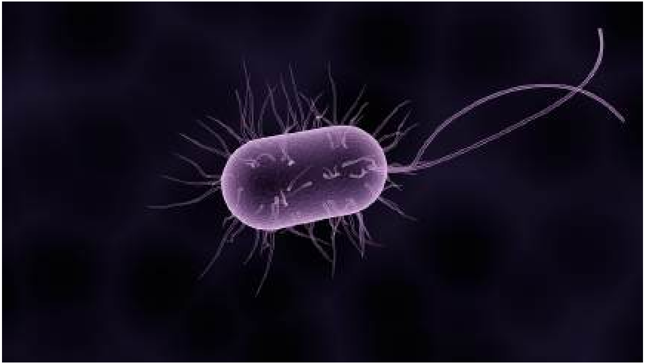
The mesoscale of the Universe ($10^{-5}m \Leftrightarrow 10^{-10}m$)



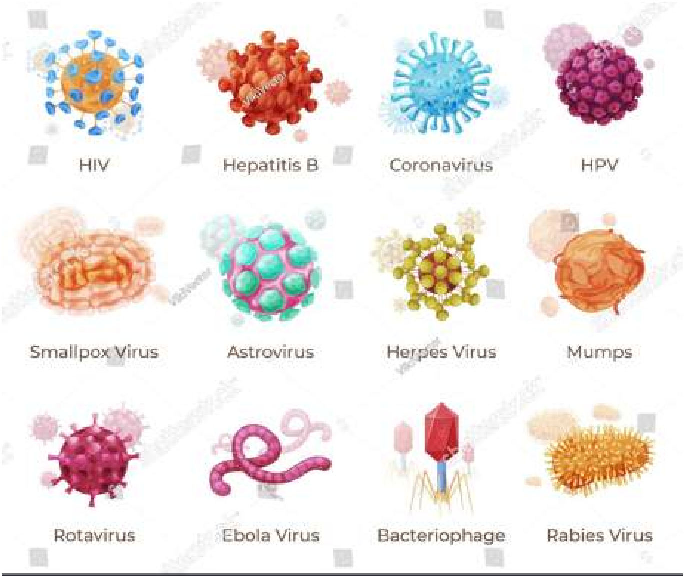
The mesoscale of the Universe ($10^{-5}m \Leftrightarrow 10^{-10}m$)



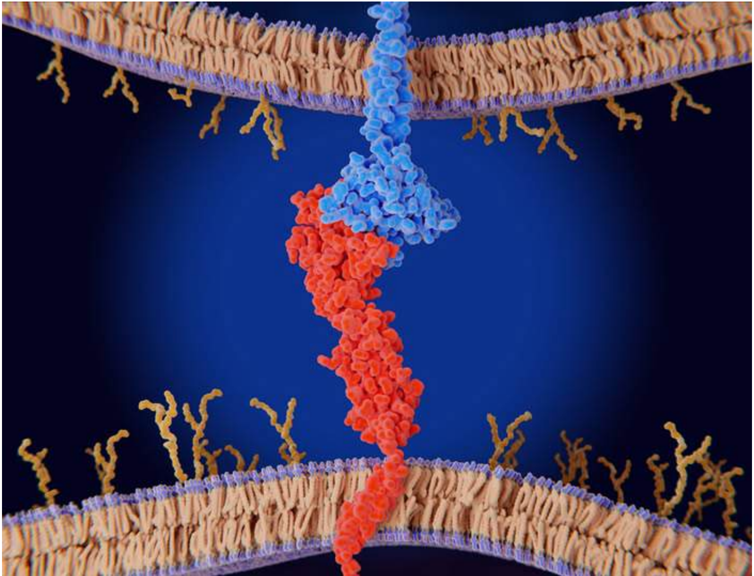
Bacteria ($0.5 \cdot 10^{-6}m \Leftrightarrow 5 \cdot 10^{-6}m$)



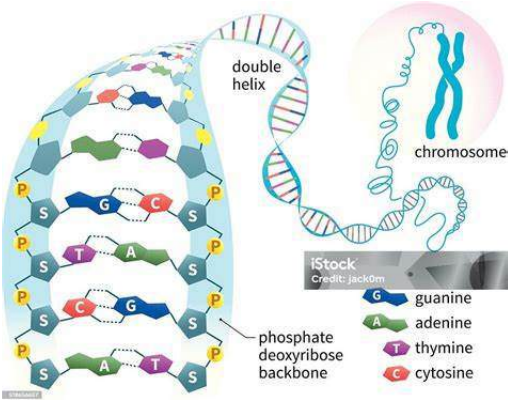
Viruses ($10^{-8}m$)



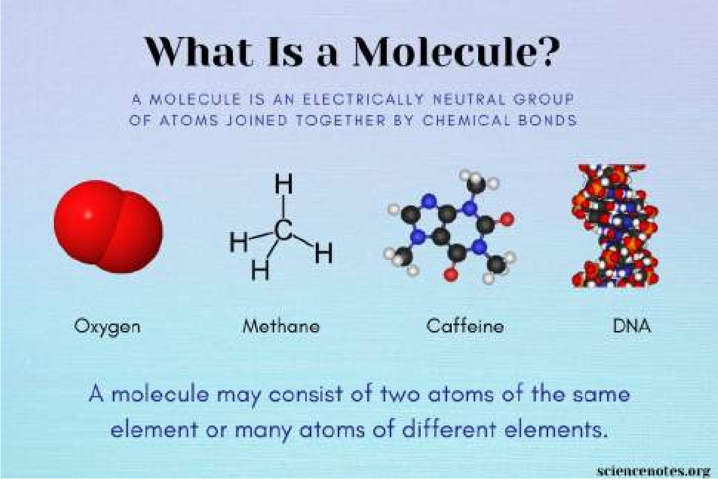
Colonies of proteins ($10^{-7}m \Leftrightarrow 10^{-9}m$)



Structure of DNA ($10^{-9}m$)



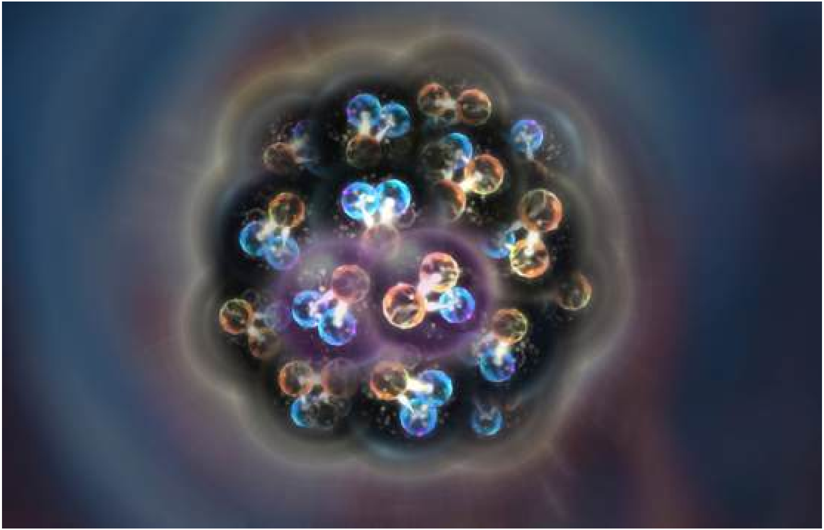
III.4. THE MICROSCALE OF THE UNIVERSE ($10^{-10}m \Leftrightarrow 10^{-35}m$)



Note. A water molecule ($2.8 \cdot 10^{-10}m$)

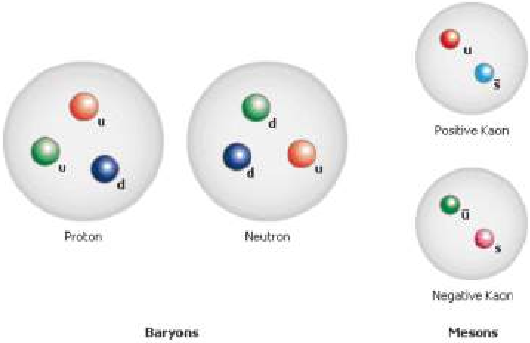
$(5 \cdot 10^{-12}m)$
Atoms (**nucleus (protons** & **generally neutrons)** & **electrons**)

Atomic nucleus ($10^{-14}m$)



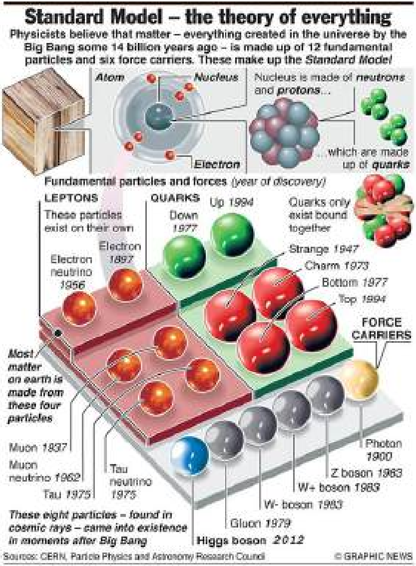
Antihyperhydrogen-4 ($5 \cdot 10^{-15}m$)

Hadrons ($10^{-15}m$)



Note. Types of meson: tetraquarks, hexaquarks & glueballs

Leptons $\cup$ Quarks & Bosons $< 10^{-15} \Leftrightarrow 10^{-23}m$



Standard Model Lagrangian

$$\mathcal{L}_{SM} = \underbrace{\frac{1}{4}W_{\mu\nu}\cdot W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G^{a}_{\mu\nu}G^{\mu\nu}_{a}}_{\text{kinetic energies and self-interactions of the gauge bosons}}$$

$$+ \underbrace{\bar{L}\gamma^{\mu}\left(i\partial_{\mu} - \frac{1}{2}g\tau\cdot W_{\mu} - \frac{1}{2}g'YB_{\mu}\right)L + \bar{R}\gamma^{\mu}\left(i\partial_{\mu} - \frac{1}{2}g'YB_{\mu}\right)R}_{\text{kinetic energies and electroweak interactions of fermions}}$$

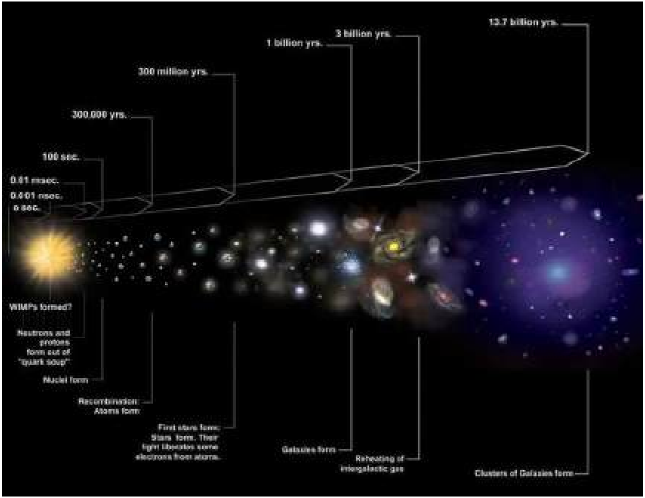
$$+ \underbrace{\frac{1}{2}\left|\left(i\partial_{\mu} - \frac{1}{2}g\tau\cdot W_{\mu} - \frac{1}{2}g'YB_{\mu}\right)\phi\right|^{2} - V(\phi)}_{W^{\pm},Z,\gamma \text{ and Higgs masses and couplings}}$$

$$+ \underbrace{g''\left(\bar{q}\gamma^{\mu}T_{a}q\right)G^{a}_{\mu}}_{\text{interactions between quarks and gluons}} + \underbrace{\left(G_{1}\bar{L}\phi R + G_{2}\bar{L}\phi_{c}R + h.c.\right)}_{\text{fermion masses and couplings to Higgs}}$$

The Planck length $1.6 \cdot 10^{-35}m$

III.5. TIME SCALES OF THE UNIVERSE ($4.3 \cdot 10^{17} s \Leftrightarrow 5.4 \cdot 10^{-44} s$)

The process of matter clustering



$5.4 \cdot 10^{-44}$ s - Planck time

10^{-25} s - an average lifetime of a W-boson W^+/W^- (weak interaction)

10^{-18} s - the smallest time interval that can be measured using modern technologies

$2 \cdot 10^{-15}$ s - the period of oscillations of the electromagnetic field of visible light

$2.2 \cdot 10^{-6}$ s - an average lifetime of a muon μ^-/μ^+

$8.85 \cdot 10^{2}$ s - an average lifetime of a neutron $n/\hat{n}$

$1.3624 \cdot 10^{17}$ s ($4.32 \cdot 10^{9}$ years) - in Hinduism, the unit of measurement of time is "Day of Brahma"

$4.35 \cdot 10^{18}$ s ($13.7 \cdot 10^{9}$ years) - the universe's existence

$> 10^{36}$ s ($> 10^{28}$ years) - e^-/e^+

$> 10^{41}$ s ($> 10^{34}$ years) - p^+/p^-

Note


Fastest phenomena in the Universe
IG:@korata33
Gravity
299,792 km/s
Neutrinos
Very close to 299,792 km/s
Expansion of outer edges of Universe
952,000 km/s
Quantum Entanglement
Instantaneous
Light
299,792 km/s

Honor to the Scientists!

Glory to Ukraïne!

Glory to Heroes!